LESSON

10 Chain Rule

We have found how to take derivatives of functions that are added, subtracted, multiplied, and divided. Next, we will cover how to work with a function that is put inside another simply by composition.

For example, it would be difficult to multiply out $f(x) = (x^3 + 10x + 4)^5$ just to take the derivative. Instead, notice that $f(x)$ looks like $g(x) = x^3 + 10x + 4$ put inside $h(x) = x^5$. Therefore, in terms of composition, $f(x) = h \circ g(x) = h(g(x))$.

The trick to differentiating composed functions is to take the derivative of the outermost layer first, while leaving the inner part alone, then multiplying by the derivative of the inside.

The Chain Rule can be stated as follows:

$$\frac{d}{dx}(h(g(x))) = h'(g(x)) \cdot g'(x)$$

If this is confusing, try stating the Chain Rule in this way:

$$\frac{d}{dx}h(\text{something}) = h'(\text{something}) \cdot \frac{d}{dx}(\text{something})$$

The Chain Rule

$$\frac{d}{dx}(h(g(x))) = h'(g(x)) \cdot g'(x) \text{ or } \frac{d}{dx}(h(\text{something})) = h'(\text{something}) \cdot \frac{d}{dx}(\text{something})$$

The usual key to figuring out what is inside and what is outside is to watch the parentheses. Imagine that the parentheses form the layers of an onion, and that you must peel (differentiate) the outermost layers before reaching the inside.

Example

Differentiate $f(x) = (x^3 + 10x + 4)^5$.

Solution

Here, $f(x) = (\text{something})^5$ where the something $= x^3 + 10x + 4$. Because $\frac{d}{dx}(x^5) = 5x^4$, the Chain Rule states the following:

$$f'(x) = 5(\text{something})^4 \cdot \frac{d}{dx}(\text{something})$$

$$f'(x) = 5(x^3 + 10x + 4)^4 \cdot \frac{d}{dx}(x^3 + 10x + 4)$$

$$f'(x) = 5(x^3 + 10x + 4)^4 \cdot (3x^2 + 10)$$

Example

Differentiate $g(x) = \sin(8x^4 + 3x^2 - 2x + 1)$.

Solution

Here, the function is essentially sin(something) where the "something" $= 8x^4 + 3x^2 - 2x + 1$. The derivative of sine is cosine, so:

$$g'(x) = \cos(\text{something}) \cdot \frac{d}{dx}(\text{something})$$

$$g'(x) = \cos(8x^4 + 3x^2 - 2x + 1) \cdot \frac{d}{dx}(8x^4 + 3x^2 - 2x + 1)$$

$$g'(x) = \cos(8x^4 + 3x^2 - 2x + 1) \cdot (32x^3 + 6x - 2)$$

Example

Differentiate $y = \cos^3(x)$.

Solution

This is tricky because of the laziness of mathematicians who like to skimp on parentheses. It might look like the "outside" function is cos(something), but it is actually $y = \cos^3(x) = (\cos(x))^3$. Thus, this function is really $(\text{something})^3$.

$$\frac{dy}{dx} = 3(\text{something})^2 \cdot \frac{d}{dx}(\text{something})$$

$$\frac{dy}{dx} = 3(\cos(x))^2 \cdot \frac{d}{dx}(\cos(x))$$

$$\frac{dy}{dx} = 3(\cos(x))^2 \cdot (-\sin(x))$$

$$\frac{dy}{dx} = -3\cos^2(x)\sin(x)$$

Example

Differentiate $y = \cos(x^3)$.

"Something" Hint

It is important that the "something" in the parentheses appear somewhere in the derivative, just as it does in the original function. If it doesn't appear, then a mistake has been made.

Solution

In this example, our function is cos(something). Because $\frac{d}{dx}(\cos(x)) = -\sin(x)$, the Chain Rule states that

$$\frac{dy}{dx} = -\sin(\text{something}) \cdot \frac{d}{dx}(\text{something})$$

$$\frac{dy}{dx} = -\sin(x^3) \cdot \frac{d}{dx}(x^3)$$

$$\frac{dy}{dx} = -\sin(x^3) \cdot 3x^2$$

And because there was an (x^3) in the original function, an (x^3) must appear in the derivative.

Example

Differentiate $h(x) = e^{5x}$.

Solution

$h(x) = e^{(\text{something})}$

so:

$$h'(x) = e^{(\text{something})} \cdot \frac{d}{dx}(\text{something})$$

$$h'(x) = e^{5x} \cdot \frac{d}{dx}(5x) = e^{5x} \cdot 5 = 5e^{5x}$$

▶ Practice

Differentiate the following.

1. $f(x) = (8x^3 + 7)^4$
2. $y = (x^2 + 8x + 9)^3$
3. $h(t) = (t^8 - 9t^3 + 3t + 2)^{10}$
4. $y = (u^5 - 3u^4 + 7)^{\frac{7}{2}}$
5. $g(x) = \sqrt{x^2 + 9x + 1}$
6. $y = \sqrt[3]{e^x + 1}$
7. $f(x) = \sin(x^2)$
8. $g(x) = \sin^2(x)$
9. $y = \ln(3t + 5)$
10. $h(x) = \cos(3x)$
11. $y = e^{(x^2)}$
12. $y = \ln(x + 1)$
13. $s(u) = \cos^5(u)$
14. $y = (\ln(x))^5$
15. $f(x) = e^x + e^{2x} + e^{3x}$
16. $y = \tan(e^x)$
17. $g(x) = \dfrac{e^x - e^{-x}}{2}$
18. $f(\theta) = \dfrac{\sin(2\theta)}{\theta}$
19. $y = xe^{2x}$
20. $f(x) = \sec(10x^2 + e^x)$

This rule is called the Chain Rule because it works in long succession when there are many layers to the function. It helps to write out the function using lots of parentheses, and then work patiently to take the derivative of each outermost layer.

Example

Differentiate $f(x) = \sin^7(e^{5x})$.

Solution

With all of its parentheses, this function is $f(x) = (\sin(e^{(5x)}))^7$. The outermost layer is "something to the seventh power," the second layer is "the sine of something," the third layer is "e raised to the something," and the last layer is $5x$. Thus:

$$f'(x) = 7(\sin(e^{(5x)}))^6 \cdot \frac{d}{dx}(\sin(e^{(5x)}))$$

$$f'(x) = 7(\sin(e^{(5x)}))^6 \cdot \cos(e^{(5x)}) \cdot \frac{d}{dx}(e^{(5x)})$$

$$f'(x) = 7(\sin(e^{(5x)}))^6 \cdot \cos(e^{(5x)}) \cdot e^{(5x)} \cdot \frac{d}{dx}(5x)$$

$$f'(x) = 7(\sin(e^{(5x)}))^6 \cdot \cos(e^{(5x)}) \cdot e^{(5x)} \cdot 5$$

$$f'(x) = 35e^{(5x)}\sin^6(e^{(5x)})\cos(e^{(5x)})$$

Example

Differentiate $y = \ln(x^3 + \tan(3x^2 + x))$.

Solution

$$\frac{dy}{dx} = \frac{1}{x^3 + \tan(3x^2 + x)} \cdot \frac{d}{dx}(x^3 + \tan(3x^2 + x))$$

$$\frac{dy}{dx} = \frac{1}{x^3 + \tan(3x^2 + x)} \cdot \left(3x^2 + \sec^2(3x^2 + x) \cdot \frac{d}{dx}(3x^2 + x)\right)$$

$$\frac{dy}{dx} = \frac{1}{x^3 + \tan(3x^2 + x)} \cdot (3x^2 + \sec^2(3x^2 + x) \cdot (6x + 1))$$

$$\frac{dy}{dx} = \frac{3x^2 + \sec^2(3x^2 + x) \cdot (6x + 1)}{x^3 + \tan(3x^2 + x)}$$

Notice once again that every part except the outermost layer (the natural logarithm) appears somewhere in the derivative.

▶ Practice

Differentiate the following.

21. $f(x) = \cos^3(8x)$

22. $y = (e^{9x^2+2x+1})^4$

23. $g(t) = \ln(\tan(e^t + 1))$

24. $y = \sin(\sin(\sin(x)))$

25. $k(u) = \sec(\ln(8u^3))$

26. $h(x) = \ln(\cos(x + e^{3x}))$